

RIPS COMPLEXES AND COVERS IN THE UNIFORM CATEGORY

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ABSTRACT. James [20] introduced uniform covering maps as an analog of covering maps in the topological category. Subsequently Berestovskii and Plaut [3] introduced a theory of covers for uniform spaces generalizing their results for topological groups [1]-[2]. Their main concepts are discrete actions and pro-discrete actions, respectively. In case of pro-discrete actions Berestovskii and Plaut provided an analog of the universal covering space and their theory works well for the so-called coverable spaces. As will be seen in Section 7, [3] generalizes only regular covering maps in topology and pro-discrete actions may not be preserved by compositions.

In this paper we redefine the uniform covering maps and we generalize pro-discrete actions using Rips complexes and the chain lifting property. We expand the concept of generalized paths of Krasinkiewicz and Minc [21]. One way to do it is by embedding X in a space with good local properties and this is done in Section 6. Another way is by systematic use of Rips complexes. In the topological category one uses paths in X originating from a base point to construct the universal covering space \tilde{X} . We use paths in Rips complexes and their homotopy classes possess a natural uniform structure, a generalization of the basic topology on \tilde{X} . Applying Rips complexes leads to a natural class of uniform spaces for which our theory of covering maps works as well as the classical one, namely the class of uniformly joinable spaces. In the case of metric continua (compact and connected metric spaces) that class is identical with pointed 1-movable spaces, a well-understood class of spaces introduced by shape theorists (see [9] or [24]). The class of pointed 1-movable continua contains all planar subcontinua (examples: Hawaiian Earring and the suspension of the Cantor set) and is preserved by continuous maps. The most notable continuum not being pointed 1-movable is the dyadic solenoid. As an application of our results we present an exposition in [7] of Prajs' [30] homogeneous curve that is path-connected but not locally connected.

CONTENTS

1. Introduction	2
2. Uniform covering maps	3
3. Generalized uniform paths	9
4. Uniform joinability	10
5. Generalized uniform covering maps	14

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5.1. Generalized uniform covering maps and uniformly joinable spaces	16
6. Generalized paths relative to spaces	19
7. Comparison to Berestovskii-Plaut uniform covers	22
References	25

1. INTRODUCTION

The aim of this paper is to develop a theory of covering maps in the uniform category via generalizations of the classical construction of universal covering spaces. For basic facts on uniform spaces we refer to [19] or [20].

In Section 2 we provide an analog of covering maps in topology adopted for the uniform category. Our definition uses local structure of the base space just as it does in topology. However, we provide a characterization of uniform covering maps via chain lifting property and that characterization is later on expanded to define generalized uniform covering maps.

How to construct universal covering space for uniform spaces X with good local properties (the so-called uniform Poincare spaces)? Let us recall briefly the construction of a simple topology (used in [31, p.82], [18, p.253], [14], [6], and [8]) on the space \tilde{X} , the space of homotopy classes (rel.endpoints) of paths in X originating from the base-point x_0 . First, one defines sets $B([\alpha], U)$ (denoted by $\langle \alpha, U \rangle$ on p.82 in [31]), where U is open in X , α joins x_0 and $\alpha(1) \in U$ as follows: $[\beta] \in B([\alpha], U)$ if and only if there is a path γ in U from $\alpha(1)$ to $\beta(1)$ such that $\beta \sim \alpha * \gamma$.

\tilde{X} equipped with the topology (which we call the **basic topology on \tilde{X}**) whose basis consists of $B([\alpha], U)$, where U is open in X , α joins x_0 and $\alpha(1) \in U$ is denoted by \hat{X} as in [4].

It turns out, for uniform spaces X , the space \tilde{X} has a natural uniform structure that generalizes the basic topology and we provide natural analogs of classical results for uniform Poincare spaces.

How to deal with spaces X whose local structure is complicated (example: the Topologist Sine Curve)? Spaces like that may not be path-connected resulting in the projection $\tilde{X} \rightarrow X$ not being surjective. The geometrical answer is to use paths in neighborhoods of X . That leads to the concept of a **generalized path** introduced by Krasinkiewicz-Minc [21]. We generalize that concept to embeddings of X in a space T with good local properties in Section 6. The resulting space $GP_T(X, x_0)$ of generalized paths has a natural uniform structure mimicking that of \tilde{X} . The advantage of embeddings is that many natural spaces are defined that way and we may apply shape-theoretical results. The disadvantage of defining universal covering spaces using only embeddings is that one has to show independence of the construction on the embedding. That is why Rips complexes are useful. In Section 3 we apply Rips complexes to define an abstract space $GP(X, x_0)$ of generalized uniform paths equipped with a natural uniform structure so that the end-point map $\pi_X: GP(X, x_0) \rightarrow X$ is uniformly continuous. As the defining characteristic of covering maps we use the Unique Path Lifts Property of any topological cover: $f: X \rightarrow Y$ is declared a **generalized uniform covering map** (see Section 5) if it has lifting and approximate uniqueness of lifts properties for both chains and generalized uniform paths. The meaning of our definition is that not only we want

the Unique Generalized Path Lifting Property but the lifting function ought to be a morphism in the uniform category.

What is the largest class of spaces for which that definition ought to work? The answer is quite simple: it is the class of **uniformly joinable spaces** X that may be characterized by the requirement of $\pi_X: GP(X, x_0) \rightarrow X$ being a generalized uniform covering map. It turns out that particular class (in case of metric continua) coincides with the class of joinable continua studied by Krasinkiewicz and Minc [21].

In Section 7 we relate our construction to that of Berestovskii and Plaut [3].

We are grateful to Conrad Plaut for a series of lectures on his work with Berestovskii. We thank Misha Levin for suggesting to provide an exposition of J.Prajs' [30] example of a homogeneous curve P that is path-connected but not locally connected (see [7]).

2. UNIFORM COVERING MAPS

We will discuss exclusively symmetric subsets E of $X \times X$ (that means $(x, y) \in E$ implies $(y, x) \in E$) and the natural notation here (see [28]) is to use $f(E)$ for the set of pairs $(f(x), f(y))$, where $f: X \rightarrow Y$ is a function. Similarly, $f^{-1}(E)$ is the set of pairs (x, y) so that $(f(x), f(y)) \in E$ if $f: X \rightarrow Y$ and $E \subset Y \times Y$.

The **ball** $B(x, E)$ **at** x **of radius** E is the set of all $y \in X$ satisfying $(x, y) \in E$.

A **uniform structure** on X is a family \mathcal{E} of symmetric subsets E of $X \times X$ (called **entourages**) that contain the diagonal of $X \times X$, form a filter (that means $E_1 \cap E_2 \in \mathcal{E}$ if $E_1, E_2 \in \mathcal{E}$ and $F_1 \in \mathcal{E}$ if $F_2 \in \mathcal{E}$ and $F_2 \subset F_1$), and every $G_1 \in \mathcal{E}$ admits $G \in \mathcal{E}$ so that $G^2 \subset G_1$ (G^2 consists of pairs $(x, z) \in X \times X$ so that there is $y \in X$ satisfying $(x, y) \in G$ and $(y, z) \in G$). A **base** \mathcal{F} of a uniform structure \mathcal{E} is a subfamily \mathcal{F} of \mathcal{E} so that for every entourage E there is a subset $F \in \mathcal{F}$ of E .

Given a decomposition of a uniform space X the most pressing issue is if it induces a natural uniform structure on the decomposition space. James [20, 2.13 on p.24] has a concept of weakly compatible relation to address that issue. For the purpose of this paper we need a different approach.

Definition 2.1. Suppose $f: X \rightarrow Y$ is a surjective function from a uniform space X . f **generates a uniform structure on** Y if the family $f(E)$, E an entourage of X , is a base of a uniform structure on Y (that particular uniform structure on Y is said to be **generated by** f). Equivalently, for each entourage E of X there is an entourage F of X such that $f(F)^2 \subset f(E)$.

Notice $f: X \rightarrow Y$ is uniformly continuous if both X and Y are uniform spaces and the uniform structure on Y is generated by f . Indeed $E \subset f^{-1}(f(E))$ for any entourage E of X .

Uniform covering maps were defined by James [20, p.112]. In this section we redefine that concept using Rips complexes and we provide a characterization of uniform covering maps in terms of chain lifting. That characterization will be very useful when generalizing uniform covering maps in Section 5.

The definition of a Rips complex for uniform structures is a straightforward generalization of Rips complexes [16, Chapter 4] for metric spaces.

Definition 2.2. Let X be a set. Given a symmetric subset E of $X \times X$ containing the diagonal define the **Rips complex** $R(X, E)$ as the subcomplex of the full complex over X whose simplices are finite subsets $F = \{x_1, \dots, x_n\}$ of X so that $F \times F \subset E$.

Notice E containing the diagonal of $X \times X$ ensures the set of vertices of $R(X, E)$ coincides with X .

Given $f: X \rightarrow Y$ and an entourage E of X notice it induces a natural simplicial map $f_E: R(X, E) \rightarrow R(Y, f(E))$ by the formula $f_E(\sum_{i=1}^n t_i \cdot x_i) = \sum_{i=1}^n t_i \cdot f(x_i)$.

Our goal is to study homotopy classes of paths in $R(X, E)$ joining two of its vertices. Since the identity function $K_w \rightarrow K_m$, K a simplicial complex, from K equipped with the CW (weak) topology to K equipped with the metric topology is a homotopy equivalence (see [24, page 302]), it does not really matter which topology we choose for $R(X, E)$. For simplicity (and to be able to use [31, Corollary 17 on p.138]), let it be the weak topology.

The simplest path in $R(X, E)$ is the **edge-path** $e(x, y)$ starting from x and ending at y so that $(x, y) \in E$.

Any path in $R(X, E)$ joining two vertices x and y can be realized, up to homotopy (see [31, Section 3.4]), as a concatenation of edge-paths. Thus, each path in $R(X, E)$ can be realized by an **E -chain** $x_1 = x, \dots, x_n = y$ such that $(x_i, x_{i+1}) \in E$ for all $i < n$. Two paths in $R(X, E)$ represented by different E -chains with the same end-points are homotopic rel. end-points if and only if one can move from one chain to the other by simplicial homotopies: a new vertex v can be added or removed from a chain if and only if v forms a simplex in $R(X, E)$ with adjacent links of a chain (see [31, Section 3.6]).

Here is our definition of covering maps in the uniform category using Rips complexes. We call a simplicial map a **simplicial covering map** if it is a topological cover.

Definition 2.3. Let X and Y be uniform spaces. $f: X \rightarrow Y$ is a **uniform covering map** if it generates the uniform structure on Y and the family \mathcal{E} of entourages E of X such that the induced map $f_E: R(X, E) \rightarrow R(Y, f(E))$ is a simplicial covering map forms a base of the uniform structure of X .

Let us characterize uniform covering maps in terms analogous to classical topological covering maps.

Definition 2.4. Let $f: X \rightarrow Y$ be a map of sets. A symmetric subset E of $X \times X$ **evenly covers** $f(E)$ if $B(x, E)$ is mapped by f bijectively onto $B(f(x), f(E))$ for all $x \in X$.

Lemma 2.5. Suppose $f: X \rightarrow Y$ is a function of uniform spaces and the uniform structure on Y is generated by f . $f: X \rightarrow Y$ is a uniform covering map if and only if X has a base of entourages E that evenly cover $f(E)$.

Proof. Suppose $f_E: R(X, E) \rightarrow R(Y, f(E))$ is a simplicial covering map. If $(x, y), (x, z) \in E$ and $f(y) = f(z)$, then the edge-path $e(f(x), f(y))$ can be lifted starting from x in two different ways unless $y = z$. That means $B(x, E)$ is mapped by f injectively into $B(f(x), f(E))$ for all $x \in X$. If $(f(x), y) \in f(E)$ we can lift the edge $e(f(x), y)$ to an edge $e(x, z)$ in $R(X, E)$. Thus $f(z) = y$ and $(x, z) \in E$.

Suppose $B(x, E)$ is mapped by f bijectively onto $B(f(x), f(E))$ for all $x \in X$. Assume $E^2 \subset F$ and F covers evenly $f(F)$. Given $x \in X$ and given a simplex Δ in $R(Y, f(E))$ containing $f(x)$ there is a unique lift Δ' of Δ containing x . Indeed, we can lift each edge of Δ emanating from $f(x)$ and the endpoints of lifts (together with x) form a lift Δ' . If $[f(x), y_1]$ and $[f(x), y_2]$ are two edges of Δ that lift to $[x, x_1]$

and $[x, x_2]$ respectively then there is an edge $[x_1, z]$ of $R(X, E)$ with $f(z) = y_2$. Now $x_2, z \in B(x_1, F)$ so $x_2 = z$. Thus every open star of a vertex in $R(Y, f(E))$ has the point inverse of the form of the disjoint union of open stars of vertices in $R(X, E)$ and f_E restricts to a homeomorphism on each of those open stars. In other words, f_E is a topological cover. \square

To show how 2.3 relates to uniform covering maps of James [20, p.112] let us define one of the main concepts of the paper.

Definition 2.6. A surjective function $f: X \rightarrow Y$ from a uniform space X has **the chain lifting property** if for any entourage E of X there is an entourage F of X such that any $f(F)$ -chain in Y starting from $f(x_0)$ can be lifted to an E -chain starting from x_0 .

A function $f: X \rightarrow Y$ from a uniform space X has **the uniqueness of chain lifts property** if for every entourage E of X there is an entourage F of X such that any two F -chains α and β satisfying $f(\alpha) = f(\beta)$ are equal if they originate from the same point.

Notice the chain lifting property is stronger than generating a uniform structure on the range (see 2.9).

James [20, p.13] defined the concept of an entourage being transverse to an equivalence relation. In the same way one can define an entourage to be transverse to a function.

Definition 2.7. Let X be a uniform space and Y be a set. An entourage E of X is **transverse** to $f: X \rightarrow Y$ if $(x, y) \in E$ and $f(x) = f(y)$ implies $x = y$.

Proposition 2.8. $f: X \rightarrow Y$ has the uniqueness of chain lifts property if and only if f has a transverse entourage.

Proof. Suppose E is an entourage of X and $F \subset E$ is chosen so that F^2 is transverse to f . Given two different F -chains $\alpha = \{x_0, \dots, x_n\}$ and $\beta = \{y_0, \dots, y_n\}$ of X originating from x_0 such that $f(\alpha) = f(\beta)$ choose the smallest i satisfying $x_i \neq y_i$. Notice $(x_i, y_i) \in F^2$ as $x_{i-1} = y_{i-1}$. Hence $x_i = y_i$ (as F^2 is transverse to f), a contradiction.

If f has the uniqueness of chain lifts property, pick an entourage $E_0 \subset X \times X$ so that any two E_0 -chains α and β are equal provided $f(\alpha) = f(\beta)$ and their origins are the same. If $f(x) = f(y)$ and $(x, y) \in E_0$, then put $\alpha = \{x, y\}$ and $\beta = \{x, x\}$. Observe $f(\alpha) = f(\beta)$. Hence $\alpha = \beta$ and E_0 is transverse to f . \square

Here is the relation of chain lifting property to the concept of uniform openness used by James [20, Definition 1.12, p.10].

Proposition 2.9. Suppose $f: X \rightarrow Y$ is a surjective function from a uniform space X to a set Y .

- a. If Y has a uniform structure making f uniformly open, then f has the chain lifting property.
- b. If f has the chain lifting property, then f generates a uniform structure on Y making f uniformly open.

Proof. a. f being uniformly open means existence, for each entourage D of X , of an entourage E of Y such that $B(f(x), E) \subset f(B(x, D))$. That condition says any pair $(y, z) \in E$ lifts to $(x, t) \in E$ if $y = f(x)$. Hence any E -chain in Y lifts to a D -chain in X . Choose an entourage F of X satisfying $f(F) \subset E$ and notice any $f(F)$ -chain lifts to a D -chain.

b. Suppose f has the chain lifting property. First, we need to show the family $\{f(E)\}_{E \in \mathcal{E}(X)}$ forms a base of entourages of Y . The only condition needed to be proved is the existence, for each entourage E of X , of an entourage F of X such that $f(F)^2 \subset f(E)$. Assume $D^2 \subset E$ and any $f(F)$ -chain in Y lifts to a D -chain in X . Suppose $(f(x), y) \in f(F)$ and $(y, z) \in f(F)$. We may choose $x_1 \in f^{-1}(y)$ so that $(x, x_1) \in D$. Now we may choose $x_2 \in f^{-1}(z)$ so that $(x_1, x_2) \in D$. Hence $(x, x_2) \in E$ and $(f(x), z) \in f(E)$.

Notice that if any $f(F)$ -chain in Y lifts to an E -chain in X , then $B(f(x), f(F)) \subset f(B(x, E))$, so f is indeed uniformly open. \square

Let us characterize covering maps in the uniform category in terms of lifting of chains.

Theorem 2.10. *Suppose $f: X \rightarrow Y$ is a function of uniform spaces and the uniform structure on Y is generated by f . f is a uniform covering map if and only if the following conditions are satisfied:*

- a) f has the chain lifting property.
- b) f has the uniqueness of chain lifts property.

Proof. Suppose f is a uniform covering map. The existence of a transverse entourage E_0 is obvious as any E such that $f_E: R(X, E) \rightarrow R(Y, f(E))$ is a simplicial covering map will do. Condition b) follows from 2.8. Also, in that case it is clear any $f(E)$ -chain in Y lifts to an E -chain in X .

Assume Conditions a) and b) are satisfied.

Given an entourage G of X define $\alpha(G)$ as the set of points $(x, y) \in G$ satisfying the following conditions:

- (1) For any $x_1 \in f^{-1}(f(x))$ there is $y_1 \in f^{-1}(f(y))$ such that $(x_1, y_1) \in G$.
- (2) For any $y_1 \in f^{-1}(f(y))$ there is $x_1 \in f^{-1}(f(x))$ such that $(x_1, y_1) \in G$.

First, observe the family $\{\alpha(G)\}_{G \in \mathcal{E}}$ forms a base of entourages of X . Indeed, given an entourage E choose an entourage $E_1 \subset E$ so that any $f(E_1)$ -chain in Y lifts to an E -chain in X . Now $E_1 \subset \alpha(E)$ as follows: given $(x, y) \in E_1$ and given $x_1 \in f^{-1}(f(x))$, one can lift the $f(E_1)$ -chain $f(x), f(y)$ to an E -chain x_1, y_1 . Similarly, if $y_1 \in f^{-1}(f(y))$, we can lift $f(y), f(x)$ to an E -chain y_1, x_1 . That means $(x, y) \in \alpha(E)$.

Second, if E_0 is an entourage of X transverse to f (provided by 2.8), then notice $\alpha(G)^2 \subset E_0$ implies f maps $B(x, \alpha(G))$ bijectively onto $B(f(x), f(\alpha(G)))$. Indeed, if $y, z \in B(x, \alpha(G))$ and $f(y) = f(z)$, then $(y, z) \in E_0$ and $y = z$. If $z \in B(f(x), f(\alpha(G)))$, there is $(x_1, y_1) \in \alpha(G)$ so that $f(x) = f(x_1)$ and $f(y_1) = z$. As $(x_1, y_1) \in \alpha(G)$ there must exist $y \in f^{-1}(f(y_1))$ satisfying $(x, y) \in G$. Notice that implies $(x, y) \in \alpha(G)$ (as $(x_1, y_1) \in \alpha(G)$) and that means any $f(\alpha(G))$ -chain lifts to an $\alpha(G)$ -chain. \square

Remark 2.11. James [20, p.111–112] defined uniform covering maps as $p: X \rightarrow B$ so that there is an entourage E transverse to p and X has a base of entourages F satisfying $R \circ F = F \circ R$, where $R = p^{-1}(\Delta B)$ is the relation on X induced by p . Unfortunately, he never added the condition that p generates the uniform structure on Y (in the language of [20] it translates to relation R being weakly compatible with the uniform structure on X). Our interpretation of Chapter 8 in [20] is that he assumes so implicitly. With that in mind our definition of uniform covering maps is equivalent to that of James. Indeed, 2.8 takes care of the uniqueness of chain lifts property and 2.9 implies that James' uniform covering maps have the chain lifting

property as a map that generates the uniform structure and satisfies $R \circ F = F \circ R$ is uniformly open. Conversely, observe that any F that evenly covers $p(F)$ satisfies $R \circ F = F \circ R$.

The most important property of covering maps in topology is that of unique lifts of paths and the fact homotopic paths have the same end-point when lifted. That leads to a quick candidate \tilde{X} for the universal cover of a pointed space (X, x_0) : it is the quotient space of the space of paths $Map((I, 0), (X, x_0))$ in X (equipped with the compact-open topology) starting from x_0 , where the equivalence relation is that of homotopy rel. end-points.

In the reminder of this section we are going to define a uniform structure on \tilde{X} mimicking the basic topology on \tilde{X} and we are going to discuss necessary and sufficient conditions for the projection $\pi_X: \tilde{X} \rightarrow X$ ($\pi_X(\alpha)$ is the end-point of α) to be a uniform covering map. It turns out, not surprisingly, those conditions involve uniform local path-connectedness and uniform semi-local simple connectedness. However, our definition of uniform local path-connectedness is much simpler than [20, Definition 8.12 on p.119] and we are unsure if the definition [20, Definition 8.13 on p.119] of uniform semi-local simple connectedness is correct as it involves existence of a base of entourages rather than just one entourage.

For each entourage E of X define E^* as the family of pairs of homotopy classes $([\alpha], [\beta])$ of paths from x_0 such that $\alpha^{-1} * \beta$ is homotopic rel. end-points to a path contained in some $B(z, E)$. The family $\{E^*\}_E$ forms a base of a uniform structure on \tilde{X} which we call the **basic uniform structure** on \tilde{X} . Notice that the projection $\pi_X: \tilde{X} \rightarrow X$ is uniformly continuous. Also, it is surjective if and only if X is path-connected.

Proposition 2.12. *If X is a path-connected uniform space, then its structure is generated by $\pi_X: \tilde{X} \rightarrow X$ if and only if for each entourage E of X there is an entourage F such that any two points in $B(x, F)$ can be connected by a path in $B(x, E)$ for any $x \in X$.*

Proof. Suppose $\pi_X: \tilde{X} \rightarrow X$ generates the structure on X . Given an entourage E of X there is an entourage F of X satisfying $F^2 \subset \pi_X(E^*)$. Suppose $y, z \in B(x, F)$. Since $(y, z) \in F^2$, there is a pair of paths $(\alpha, \beta) \in E^*$ so that α joins x_0 to y and β joins x_0 to z . Thus $\alpha^{-1} * \beta$ is homotopic rel. end-points to a path contained in some $B(w, E) \subset B(x, E^3)$.

Suppose for each entourage E of X there is an entourage F such that any two points in $B(x, F)$ can be connected by a path in $B(x, E)$ for any $x \in X$. Given $(y, z) \in F$ choose a path α contained in $B(y, E)$ joining y to z . Choose a path β from x_0 to y and observe $(\beta, \beta * \alpha) \in E^*$, $\pi_X(\beta, \beta * \alpha) = (y, z)$ which proves $F \subset \pi_X(E^*)$ and π_X generates the structure on X . \square

Call a space satisfying the conditions in 2.12 **uniformly locally path-connected** (see [28] or [3]).

Lemma 2.13. *If X is uniformly locally path-connected, then for every entourage E of X there is an entourage $F \subset E$ such that $B(x, F)$ is path-connected for every $x \in X$.*

Proof. Let H be an entourage of X such that for any $x \in X$, any two points in $B(x, H)$ can be joined by a path in $B(x, E)$. Define F to be all $(x, y) \in E$ such that x and y can be joined by a path in some $B(z, H)$. Notice $H \subset F$ so that F is

an entourage. Let $x \in X$ and $y, z \in B(x, F)$. Then there is a path α joining y to x in some $B(z_1, E)$ and a path β joining x to z in some $B(z_2, E)$. Notice that $\alpha * \beta$ is contained in $B(x, F)$. \square

Proposition 2.14. *Suppose $f : X \rightarrow Y$ is a uniform covering map and $g : Z \rightarrow Y$ is uniformly continuous. Suppose X, Y , and Z are path-connected and uniformly locally path-connected. Let $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$ with $f(x_0) = g(z_0) = y_0$. Then there is a unique uniformly continuous lift $\tilde{g} : Z \rightarrow X$ of g with $\tilde{g}(z_0) = x_0$ if and only if $g_*(\pi_1(Z, z_0)) \subset f_*(\pi_1(X, x_0))$. Further, if g is a uniform covering map then so is \tilde{g} .*

Proof. From the corresponding theorem in the topological category we have the forward direction. Also from the topological theorem there is a unique lift \tilde{g} with $\tilde{g}(z_0) = x_0$ defined by letting $\tilde{g}(z)$ be the endpoint of the lift of $g \circ \alpha$ starting at x_0 where α is a path from z_0 to z [26]. We show that \tilde{g} is uniformly continuous. Let E be an entourage of X evenly covering $f(E)$ and F be an entourage of Z such that $B(z, F)$ is path-connected for each $z \in Z$ and $g(F) \subset f(E)$. Let $(x, y) \in F$. Take a path α from z_0 to x and a path β from x to y that is contained in $B(x, F)$. Lift $g \circ \alpha$ to a path $\tilde{\alpha}$ from x_0 to $\tilde{g}(x)$. Since $g \circ \beta$ is contained in $B(g(x), f(E)) = B(f(\tilde{g}(x)), f(E))$, $\gamma = (f|_{B(\tilde{g}(x), E)})^{-1} \circ g \circ \beta$ is a path starting at $\tilde{g}(x)$ that is contained in $B(\tilde{g}(x), E)$. But $\tilde{\alpha} * \gamma$ is a lift of $g \circ (\alpha * \beta)$ starting at x_0 so $\gamma(1) = \tilde{g}(y)$ and we have $(\tilde{g}(x), \tilde{g}(y)) \subset E$.

Now suppose g is a uniform covering map. Let us first show that \tilde{g} is surjective. If $x \in X$, take a path α from x_0 to x and lift $f \circ \alpha$ to a path $\tilde{\alpha}$ in Z starting at z_0 . Then $\tilde{g}(\tilde{\alpha}(1)) = x$.

Now let us see that \tilde{g} generates the uniform structure of X . Suppose E is an entourage of Z evenly covering $g(E)$ and let F be an entourage of X that evenly covers $f(F)$ and has $f(F) \subset g(E)$. Finally, let $G \subset F$ be an entourage so that for every $x \in X$, any two points in $B(x, G)$ can be joined by a path contained in $B(x, F)$. Suppose $(x, y) \in G$. Take a path α from x_0 to x and a path β from x to y that is contained in $B(x, F)$. Lift $f \circ \alpha$ to a path $\tilde{\alpha}$ in Z starting at z_0 and set $x' = \tilde{\alpha}(1)$. Set $\gamma = (g|_{B(x', E)})^{-1} \circ f \circ \beta$ and notice that $\tilde{\alpha} * \gamma$ is a path from z_0 to some $y' \in B(x', E)$. Then $\tilde{g}(x', y') = (x, y)$ so we have $G \subset \tilde{g}(E)$ and \tilde{g} the uniform structure of X .

Finally, put $H = E \cap \tilde{g}^{-1}(G)$ where E and G are as above and let us see that H evenly covers $\tilde{g}(H)$. Let $z \in Z$ and suppose $x, y \in B(z, H)$ with $\tilde{g}(x) = \tilde{g}(y)$. Then $g(x) = g(y)$ so $x = y$ since E evenly covers $g(E)$. Now suppose $y \in B(\tilde{g}(z), \tilde{g}(H))$. Take a path α in Z from z_0 to z and let $\tilde{\alpha}$ be the lift of $g \circ \alpha$. Now take a path β from $\tilde{g}(z)$ to y that is contained in $B(\tilde{g}(z), F)$. Set $\gamma = (g|_{B(z, E)})^{-1} \circ f \circ \beta$ and notice $\tilde{\alpha} * \gamma$ is a path from z_0 to some $y' \in B(z, E)$. Then $\tilde{g}(y') = y$ and $y' \in B(z, H)$. \square

When is $\pi_X : \tilde{X} \rightarrow X$ a uniform covering map?

Proposition 2.15. *Let X be a path-connected uniform space. Suppose E is an entourage of X and $x \in X$. If $(E^*)^2$ is transverse to π_X , then every loop in $B(x, E)$ at x is null-homotopic in X .*

Proof. Notice balls $B(\alpha, E^*)$, $\alpha \in \pi_X^{-1}(x)$, are mutually disjoint. Suppose γ is a loop in $B(x, E)$ at x . Choose α joining x_0 to x and notice $(\alpha * \gamma, \alpha) \in E^*$. Since $\pi_X(\alpha) = x = \pi_X(\alpha * \gamma)$, $\alpha * \gamma$ is homotopic rel. end-points to α in X and γ is null-homotopic in X . \square

2.12 and 2.15 lead to the concept of a **uniform Poincare space** X (compare [3]), a space that is path-connected, uniformly locally path-connected, and **uniformly semi-locally simply connected** (that means the existence of an entourage F such that all loops in $B(x, F)$ at x are null-homotopic in X for all $x \in X$).

Theorem 2.16. $\pi_X: \tilde{X} \rightarrow X$ is a uniform covering map if and only if X is a uniform Poincare space.

Proof. If π_X is a uniform covering map, X must be uniformly locally path-connected by 2.12 and uniformly semi-locally simply connected by 2.15.

Suppose X is a uniform Poincare space. By 2.12 π_X generates the uniform structure of X . Let F be an entourage of X and let E be an entourage of X such that loops in $B(x, E)$ at x are null-homotopic in X . Let G be an entourage with $G^2 \subset F \cap E$ and $H \subset G$ be an entourage such that all balls $B(x, H)$ are path-connected (use 2.13).

Let us show that H^* evenly covers $\pi_X(H^*)$. Let $\alpha \in \tilde{X}$ and $\beta, \gamma \in B(\alpha, H^*)$ with $\pi_X(\beta) = \pi_X(\gamma) = x$ for some $x \in X$. Notice $\beta^{-1} * \gamma$ is homotopic rel. end-points to a path contained in $B(x, H^2) \subset B(x, E)$ so it is null homotopic. Therefore $\beta \sim \gamma$. Now let $y \in B(\pi_X(\alpha), \pi_X(H^*))$, so $y = \pi_X(\beta)$ and $\pi_X(\alpha) = \pi_X(\alpha')$ for some $(\beta, \alpha') \in H^*$. Then $y, \pi_X(\alpha) \in B(z, H)$ for some $z \in X$ so there is a path γ joining them that is contained in $B(z, H)$. Notice $\pi_X(\alpha * \gamma) = y$ and $(\alpha, \alpha * \gamma) \in H^*$. \square

3. GENERALIZED UNIFORM PATHS

How to adjust the above construction of \tilde{X} for spaces with bad local properties? A good way is to approximate X by its Rips complexes. An alternative way is to embed X in a space with good local properties and use paths there (see Section 6).

First, we will extend the concept of paths being homotopic rel. end-points.

Definition 3.1. Two paths c and d in $R(X, E)$ with endpoints in X are **E -homotopic** provided the following conditions are satisfied:

1. The initial points x_c and x_d and the terminal points y_c and y_d of the paths c and d satisfy $(x_c, x_d), (y_c, y_d) \in E$;
2. c is homotopic in $R(X, E)$ rel. end-points to the concatenation $e(x_c, x_d) * d * e(y_d, y_c)$.

Notice the relation of being E -homotopic is symmetric and coincides with usual homotopy of paths rel. end-points in $R(X, E)$ if the end-points of paths are the same.

Given a uniform space X one can consider the space $GP(X)$ of generalized paths in X . A **generalized path** is a collection $\{[c_E]\}_E$ of homotopy classes of paths $[c_E]$ in $R(X, E)$ joining fixed $x \in X$ to $y \in X$ such that for all entourages $F \subset E$, c_F is homotopic to c_E in $R(X, E)$ rel. end-points.

A generalized path $c = \{[c_E]\}_E$ is called **F -short** if its end-points x and y satisfy $(x, y) \in F$ and $[c_F]$ is the homotopy class of the edge-path $e(x, y)$ in $R(X, F)$. In other words, c is F -short if c_F is F -homotopic to the constant path at the origin of c .

We equip $GP(X)$ with a natural uniform structure: a base of entourages of $GP(X)$ is the family F^* consisting of all pairs (c, d) of generalized paths $c = \{[c_E]\}_E$ and $d = \{[d_E]\}_E$ such that c_F is F -homotopic to d_F .

If two generalized paths c and d have the same initial point (or the same terminal point), then $(c, d) \in F^*$ if and only if $c^{-1} * d$ is F -short ($c * d^{-1}$ is F -short, respectively).

The **projection** $\pi_X: GP(X) \rightarrow X$ assigns to each generalized path its end-point. Notice π_X is uniformly continuous as $(c, d) \in F^*$ implies $(\pi_X(c), \pi_X(d)) \in F$.

Given a uniform morphism $f: X \rightarrow Y$ it induces a function $\tilde{f}: GP(X) \rightarrow GP(Y)$ as follows: Let $c = \{[c_E]\}_E$ be any generalized path of X and F be any entourage of Y . Put $E = f^{-1}(F)$ and define $\tilde{f}(c) = \{[f_E(c_E)]\}_F$. Notice that all paths $f_H(c_H)$, $H \subset E$, are homotopic rel. end-points to $f_E(c_E)$ so that \tilde{f} is well-defined. Also notice that \tilde{f} is uniformly continuous as for any entourages E of X and F of Y the inclusion $E \subset f^{-1}(F)$ implies $\tilde{f}(E^*) \subset F^*$.

Given a pointed uniform space (X, x_0) one can consider the space $GP(X, x_0)$ of generalized paths in X originating from x_0 with the uniform structure induced from $GP(X)$. Any pointed uniformly continuous function $f: (X, x_0) \rightarrow (Y, y_0)$ induces a uniformly continuous $\tilde{f}: GP(X, x_0) \rightarrow GP(Y, y_0)$.

In case of $(X, x_0) = (I, 0)$ being the pointed unit interval the space $GP(I, 0)$ is naturally identical with I as for any $t \in I$ there is only one generalized uniform path from 0 to t (a generalization of this observation is Corollary 5.11). Therefore every ordinary path in X from x_0 to x induces naturally a generalized uniform path which we will usually denote by the same letter.

4. UNIFORM JOINABILITY

Connectivity and path connectivity can be generalized to the uniform category in several ways. First, the concept of **chain connectivity** of X (see [28] or [3]) that is equivalent to **uniform connectivity** of James [20, Definition 1.5 on p.7] can be formulated as connectivity of all its Rips complexes.

Here is a generalization of path-connectivity.

Definition 4.1. X is **joinable** if any of its two points can be joined by a generalized uniform path.

Obviously, any X such that the underlying topological space is path-connected, is joinable.

The following is an elementary exercise:

Proposition 4.2. *If X is a uniform space, then the following conditions are equivalent:*

- a. X is joinable,
- b. $\pi_X: GP(X, x_0) \rightarrow X$ is surjective for each $x_0 \in X$,
- c. $\pi_X: GP(X, x_0) \rightarrow X$ is surjective for some $x_0 \in X$.

Definition 4.3. X is **uniformly joinable** if for each entourage E of X there is an entourage F such that any pair $(x, y) \in F$ can be joined by a generalized path $\{[c_H]\}_H \in GP(X)$ that is E -short.

Notice that any uniformly locally path-connected X is uniformly joinable. Those include inner-metric spaces (in particular, geodesic spaces) and Peano continua.

Proposition 4.4. *If $f: X \rightarrow Y$ generates the uniform structure of Y and X is uniformly joinable, then Y is uniformly joinable.*

Proof. Given an entourage E of Y pick an entourage $F \subset G = f^{-1}(E)$ of X so that for any pair $(x, y) \in F$ there is a generalized path $c(x, y)$ joining x and y that is G -short. Suppose $(x', y') \in f(F)$. Pick a pair $(x, y) \in F$ satisfying $f(x, y) = (x', y')$ and observe $\tilde{f}(c(x, y))$ is a generalized path in Y joining x' and y' whose E -th term is $e(x', y')$ in $R(Y, E)$. \square

Proposition 4.5. *If X is uniformly joinable and chain connected, then it is joinable.*

Proof. Given an entourage E of X pick an entourage F of X so that any pair $(y, z) \in F$ can be connected by a generalized path $c(y, z)$ that is E -short. Since x_0 and x_1 can be connected by an F -chain, we can replace each link of that chain by a generalized path and obtain a generalized path d from x_0 to x_1 . \square

Proposition 4.6. *If X is chain connected, then the following conditions are equivalent:*

- a. X is uniformly joinable,
- b. $\pi_X: GP(X, x_0) \rightarrow X$ generates the uniform structure of X for each $x_0 \in X$,
- c. $\pi_X: GP(X, x_0) \rightarrow X$ generates the uniform structure of X for some $x_0 \in X$.

Proof. a) \implies b). π_X is surjective by 4.5 and 4.2. Given an entourage E of X pick an entourage F of X so that any pair $(y, z) \in F$ can be connected by a generalized path $c(y, z)$ that is E -short. Let d be a generalized uniform path from x_0 to y . Now, $d * c(y, z)$ is a generalized path in X so that $(d, d * c(y, z)) \in E^*$. Since $\pi_X(d) = y$ and $\pi_X(d * c(y, z)) = z$, we obtain $F \subset \pi_X(E^*)$ which proves π_X generates the uniform structure of X .

c) \implies a). If π_X generates the uniform structure of X , then for each entourage E of X there is an entourage $F \subset E$ of X such that $F \subset \pi_X(E^*)$. That means for any pair $(x, y) \in F$ there is $(c, d) \in E^*$ with $x = \pi_X(c)$ and $y = \pi_X(d)$. Notice $e = c^{-1}d$ is a generalized E -short path from x to y . \square

Definition 4.7. Suppose X is a uniform space and $x_0 \in X$. By the **uniform fundamental pro-group** $\text{pro-}\pi_1(X, x_0)$ we mean the inverse system of groups $\{\pi_1(R(X, E), x_0)\}_E$.

The **uniform fundamental group** $\tilde{\pi}_1(X, x_0)$ is the inverse limit of $\text{pro-}\pi_1(X, x_0)$ which is identical with the group of generalized loops of X at x_0 . Notice $\tilde{\pi}_1(X, x_0)$ inherits a uniform structure from $GP(X, x_0)$, so it is actually a topological group.

Recall an inverse system $\{G_a\}_{a \in A}$ of groups satisfies the **Mittag-Leffler condition** (see [9, p.77] or [24, p.165]) if for every $a \in A$ there is $b > a$ such that for any $c > b$ the image of $G_b \rightarrow G_a$ is contained in the image of $G_c \rightarrow G_a$ (that implies those images are actually equal). In particular, an inverse system $\{G_a\}_{a \in A}$ of groups is **trivial** if for every $a \in A$ there is $b > a$ such that the image of $G_b \rightarrow G_a$ is trivial.

As noted in [9, Proposition 6.1.2] an inverse system of groups satisfies the Mittag-Leffler condition if and only if it is movable in the category of pro-sets. Therefore it makes sense to consider a condition equivalent to uniform movability (see [24, p.160]) of a pro-group in the category of pro-sets.

Definition 4.8. An inverse system $\{G_a\}_{a \in A}$ of groups with inverse limit G satisfies the **strong Mittag-Leffler condition** if for every $a \in A$ there is $b > a$ such that

the image of $G \rightarrow G_a$ contains the image of $G_b \rightarrow G_a$ (that implies those images are actually equal).

Remark 4.9. If the index set A has a countable cofinal set B (that means for any $a \in A$ there is $b \in B$ with $b \geq a$), then $\{G_a\}_{a \in A}$ satisfying Mittag-Leffler condition implies it satisfying the strong Mittag-Leffler condition (use [24, Theorem 4 on p.163]).

Theorem 4.10. *If X is uniformly joinable then $\text{pro-}\pi_1(X, x)$ satisfies the strong Mittag-Leffler condition for each $x \in X$.*

Proof. Fix $x \in X$. Given an entourage E of X pick an entourage $F \subset E$ with the property that any pair of points $(y, z) \in F$ can be connected by a generalized path $c(y, z)$ so that $c(y, z)_E$ is the homotopy class of the edge $e(y, z)$ in $R(X, E)$. Suppose α is a loop at x in $R(X, F)$. Represent that loop as an F -chain $x = x_1, \dots, x_n = x$ and replace each edge $e(x_i, x_{i+1})$ by $c(x_i, x_{i+1})$. The result is a generalized loop γ at x so that $[\gamma_E] = [\alpha]$ in $R(X, E)$. Notice that γ_H , $H \subset F$, represents an element of $\pi_1(R(X, H), x)$ whose image in $\pi_1(R(X, E), x)$ is the same as $[\alpha]$. \square

Theorem 4.11. *Suppose X is a joinable uniform space. If $\text{pro-}\pi_1(X, x_0)$ satisfies the strong Mittag-Leffler condition for some $x_0 \in X$, then X is uniformly joinable.*

Proof. Given an entourage E of X choose an entourage $F \subset E$ with the property $\text{im}(\pi_1(R(X, F), x_0) \rightarrow \pi_1(R(X, E), x_0)) \subset \text{im}(\check{\pi}_1(X, x_0) \rightarrow \pi_1(R(X, E), x_0))$. If $(x, y) \in F$ choose a generalized path $c(x)$ from x_0 to x and choose a generalized path $c(y)$ from x_0 to y . The loop $c(x)_F * e(x, y) * c(y)_F^{-1}$ in $R(X, F)$ equals d_E in $R(X, E)$ for some generalized loop d at x_0 . Consider $c = c(x)^{-1} * d * c(y)$ and notice $c_E = e(x, y)$ which proves X is uniformly joinable. \square

Proposition 4.12. *If X is a chain connected uniform space, then any of the following conditions implies that X is joinable:*

- (1) $\text{pro-}\pi_1(X, x_0)$ is trivial for some $x_0 \in X$;
- (2) X has a countable base of entourages and $\varprojlim (\text{pro-}\pi_1(X, x_0)) = 0$ for some $x_0 \in X$.

Proof. (1) Consider the Rips complex $R(\mathcal{E})$ of the family of entourages of X . A simplex in $R(\mathcal{E})$ is a finite set $\Delta = \{E_1, \dots, E_k\}$ of entourages of X such that for any pair $i, j \leq k$ either $E_i \subset E_j$ or $E_j \subset E_i$. Each Δ has a minimal vertex $m(\Delta)$ defined as $\bigcap_{i=1}^k E_i$. By induction on the number of vertices of Δ find an entourage $a(\Delta) \subset m(\Delta)$ such that the inclusion-induced homomorphism $\pi_1(R(X, a(\Delta)), x_0) \rightarrow \pi_1(R(X, m(\Delta)), x_0)$ is trivial and the function $a(\Delta)$ is monotone (i.e. $a(\Delta) \subset a(\Delta')$ whenever Δ' is a face of Δ).

Fix a point $x \in X$. Then any two paths from x_0 to x in $R(X, a(E))$ are homotopic in $R(X, E)$. Let c_E be such a path. Then $\{[c_E]\}_E$ is a generalized path from x_0 to x . Indeed, if $F \subset E$, then for $\Delta = \{F, E\}$ one has $a(\Delta) \subset a(F) \subset F$ and $a(\Delta) \subset a(E) \subset E$, so a path in $R(X, a(\Delta))$ from x_0 to x is homotopic rel. endpoints to both c_E and c_F in $R(X, E)$.

(2) Let E_n be a base of entourages of the uniform structure on X . We can assume $E_{i+1} \subset E_i$ for all i since $\bigcap_{i=1}^n E_i$ is also a base. Put $G_n = \pi_1(R(X, E_n), x_0)$.

Recall (see [24]) that $\varprojlim \{G_n\} = 0$ means existence, for each sequence $g_n \in G_n$, of a sequence $h_n \in G_n$ such that $g_k = h_k \cdot h_{k+1}^{-1}$ in G_k for all $k \geq 1$. If each G_n is countable, that condition is equivalent to $\{G_n\}$ satisfying the Mittag-Leffler condition (see [9, p.78]).

Given $x \in X$ choose, for each $n \geq 1$, a path p_n in $R(X, E_n)$ from x_0 to x . Put $g_n = [p_n * p_{n+1}^{-1}]$ and choose loops h_n at x_0 so that $h_k * h_{k+1}^{-1}$ is homotopic rel. x_0 to g_k in $R(X, E_k)$. Put $c_k = p_k^{-1} * h_k$ for $k \geq 1$. For each E choose $E_k \subset E$ and set $c_E = c_k$. We then have a generalized path in X from x_0 to x . \square

Proposition 4.13. *If X is a uniformly joinable uniform space, then $\text{pro-}\pi_1(GP(X, x_0), y_0)$ is trivial for any $x_0 \in X$, where y_0 is the constant generalized path at x_0 in X .*

Proof. Given an entourage E of X choose an entourage $F \subset E$ such that any two points $(x, y) \in F$ can be connected by a generalized path $c(x, y)$ that is E -short. Take any loop in $R(GP(X), F^*)$ based at y_0 and represent it as a sequence $y_0, \dots, y_k = y_0$ of generalized paths in X . Let x_i be the endpoint of y_i . Then x_0, \dots, x_k is an F -chain that is F -homotopic to $(y_1 * y_1^{-1} * y_2 * y_2^{-1} * \dots * y_{k-1} * y_{k-1}^{-1})_F$ and is therefore null-homotopic via a finite sequence of simplicial homotopies in $R(X, F)$. We wish to mimick those simplicial homotopies in $R(GP(X), E^*)$. At each stage of the homotopy we will have an E^* -chain in $GP(X, x_0)$ such that the endpoints of the links of the chain form an F -chain in X . In case of a vertex reduction, say x_i , the sequence $y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_k$ is an E^* -chain since $(y_{i-1}^{-1} * y_{i+1})_E$ is homotopic to $(y_{i-1}^{-1} * y_i * y_i^{-1} * y_{i+1})_E$ which in turn is E -homotopic to $e(x_{i-1}, x_i) * e(x_i, x_{i+1})$ and the simplex $[x_{i-1}, x_i, x_{i+1}] \in R(X, E)$. Also the endpoints $x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k$ form an F -chain. In the case of inserting a new vertex z between x_i and x_{i+1} we create a new sequence $y_0, \dots, y_i, y_i * c(x_i, z), y_{i+1}, \dots, y_k$. This sequence is an E^* -chain since $((y_i * c(x_i, z))^{-1} * y_{i+1})_E$ is E -homotopic to $e(z, x_i) * e(x_i, x_{i+1})$ and the simplex $[z, x_i, x_{i+1}] \in R(X, E)$. Again, the endpoints form an F -chain. \square

Corollary 4.14. *If X is uniformly joinable, then $GP(X, x_0)$ is chain connected and uniformly joinable for any $x_0 \in X$.*

Proof. Put $Y = GP(X, x_0)$ and let y_0 be the constant generalized path at x_0 in X . Given an entourage E of X choose an entourage $F \subset E$ such that any pair $(x, y) \in F$ can be connected by a generalized path $c(x, y)$ whose E -th term is $e(x, y)$. If c is an element of $GP(X, x_0)$ look at c_F and pick its simplicial representative, an edge-path x_0, x_1, \dots, x_n . Let d be the concatenation of $c(x_i, x_{i+1})$, $i = 0, \dots, n-1$. Put $e = c * d^{-1}$ and notice $(y_0, e) \in E^*$. Now the sequence $y_0, y_1 = e, y_2 = e * c(x_0, x_1), \dots, y_{n+1} = c$ (here $y_{i+1} = y_i * c(x_{i-1}, x_i)$) joins y_0 and c so that $(y_i, y_{i+1}) \in E^*$ for all i . Thus Y is chain connected. Application of 4.13, 4.12, and 4.11 completes the proof. \square

Corollary 4.15. *If X is uniformly joinable, then for any $x_0 \in X$ the projection $\pi_{GP(X, x_0)}: GP(GP(X, x_0), c) \rightarrow GP(X, x_0)$ is a uniform equivalence for any $c \in GP(X, x_0)$.*

Proof. By 4.14 the space $Y = GP(X, x_0)$ is chain connected and uniformly joinable. By 4.6 $\pi_Y: GP(Y, c) \rightarrow Y$ generates the uniform structure of Y and by 4.13 it is injective. Therefore π_Y is a uniform equivalence. \square

5. GENERALIZED UNIFORM COVERING MAPS

We define generalized uniform covering maps by weakening conditions of 2.10 (for relations between uniform covering maps and generalized uniform covering maps via inverse limits see [22]).

Definition 5.1. A **generalized uniform covering map** is a function $f: X \rightarrow Y$ of uniform spaces generating the uniform structure of Y and satisfying the following conditions:

- GP1. (**Generalized Path Lifting Property**) Every generalized uniform path in Y at $f(x_0)$ lifts to a generalized uniform path in X at x_0 .
- GP2. (**Approximate Uniqueness of Generalized Path Lifts Property**) For any entourage E of X there is an entourage F of X such that any two generalized uniform paths α and β in X with a common origin must be E -homotopic if $f(\alpha)$ and $f(\beta)$ are $f(F)$ -homotopic.
- C1. f has the chain lifting property.
- C2. For any entourage E of X there is an entourage F of X such that any two F -chains α and β with a common origin must be E -homotopic if $f(\alpha)$ and $f(\beta)$ are $f(F)$ -homotopic.

Notice that Conditions C1 and C2 are discrete versions of Conditions GP1 and GP2, respectively.

Before analyzing interdependence of Conditions GP1-2 and C1-2 let us explain the meaning of Conditions GP1-2.

Proposition 5.2. *Suppose X and Y are Hausdorff uniform spaces and $f: X \rightarrow Y$ generates the uniform structure of Y .*

- (1) *If f satisfies Conditions GP1-2, then $\tilde{f}: GP(X, x_0) \rightarrow GP(Y, f(x_0))$ is a uniform equivalence for each $x_0 \in X$.*
- (2) *If X is joinable and $\tilde{f}: GP(X, x_1) \rightarrow GP(Y, f(x_1))$ is a uniform equivalence for some $x_1 \in X$, then f satisfies Conditions GP1-2.*

Proof. (1) Condition GP1 of 5.1 says \tilde{f} is surjective and Condition GP2 of 5.1 implies \tilde{f} is both injective and generates the uniform structure of $GP(Y, f(x_0))$. Indeed, if $\tilde{f}(\alpha) = \tilde{f}(\beta)$, then α is E -homotopic to β for all entourages E of X . Hence their end-points coincide and $\alpha = \beta$. Condition GP2 means (provided GP1 holds) $F^* \subset \tilde{f}(E^*)$, so \tilde{f} generates the uniform structure of $GP(Y, f(x_0))$.

(2) Suppose α is a generalized uniform path in Y starting at $f(x_0)$. Choose a generalized uniform path γ from x_1 to x_0 and let β be a generalized uniform path from x_1 satisfying $\tilde{f}(\beta) = \tilde{f}(\gamma) * \alpha$. Put $\sigma = \gamma^{-1} * \beta$ and observe $\tilde{f}(\sigma) = \alpha$. That proves GP1.

Choose an entourage F of Y so that $F^* \subset \tilde{f}(E^*)$ (such F exists as \tilde{f} is a uniform equivalence). Suppose α and β are two generalized uniform paths at x_0 such that $f(\alpha)$ is F -homotopic to $f(\beta)$. Choose a generalized uniform path γ from x_1 to x_0 and observe $(\tilde{f}(\gamma * \alpha), \tilde{f}(\gamma * \beta)) \in F^*$. That implies there are two generalized uniform paths $(\alpha_1, \beta_1) \in E^*$ starting from x_1 so that $\tilde{f}(\alpha_1) = \tilde{f}(\gamma * \alpha)$ and $\tilde{f}(\beta_1) = \tilde{f}(\gamma * \beta)$. Due to \tilde{f} being injective, $\alpha_1 = \gamma * \alpha$ and $\beta_1 = \gamma * \beta$. Now $\alpha^{-1} * \beta = \alpha_1^{-1} * \beta_1$ is E -short and Condition GP2 holds. \square

Corollary 5.3. *Suppose $f: X \rightarrow Y$ is a generalized uniform covering map and $y_0 = f(x_0)$. If Z is joinable, then for every uniformly continuous $g: Z \rightarrow Y$ with*

$g(z_0) = y_0$ there is at most one uniformly continuous lift $h: Z \rightarrow X$ of g satisfying $h(z_0) = x_0$.

Proof. Given $z \in Z$ pick a generalized path c from z_0 to z . Since $\tilde{f}(\tilde{h}(c)) = \tilde{g}(c)$, the generalized path $\tilde{h}(c)$ is uniquely determined. Hence its end-point $h(z)$ is uniquely determined as well. \square

The following result has an easy proof, so it is left to the reader.

Proposition 5.4. *Suppose $f: X_1 \rightarrow X_2$ and $g: X_2 \rightarrow X_3$ generate the uniform structure of their ranges. If f and g are generalized uniform covering maps, then so is the composition $g \circ f$.*

Our next objective is to replace Condition C2 by Approximate Uniqueness of Chain Lifts Property as it is closer to the uniqueness of lifts property in our definition of uniform covering maps.

Lemma 5.5. *Given a function $f: X \rightarrow Y$ from a uniform space X consider the following conditions:*

- C2. *For any entourage E of X there is an entourage F of X such that any two F -chains α and β with a common origin must be E -homotopic if $f(\alpha)$ and $f(\beta)$ are $f(F)$ -homotopic.*
- C3. (**Approximate Uniqueness of Chain Lifts Property**) *For any entourage E of X there is an entourage F of X such that any two F -chains in X starting from x_0 are E -close if their images are identical.*

If f has the chain lifting property, then C2 and C3 are equivalent.

Proof. C2 \implies C3. If $f(\alpha) = f(\beta)$, then they are clearly $f(F)$ -homotopic. Hence α is E -homotopic to β . In particular, their end-points are E -close. The same argument works of subchains of α and β with the same number of links, so α is E -close to β .

C3 \implies C2. Given an entourage E of X choose an entourage E_a satisfying $E_a^2 \subset E$. Now choose an entourage E_b of X so that any two E_b^2 -chains must be E_a -close if their images are identical. Pick an entourage $F \subset E_b$ of X such that any $f(F)$ -chain in Y lifts to an E_b -chain in X .

Consider two F -chains α and β starting from x_0 with common end-point such that $f(\alpha)$ and $f(\beta)$ are $f(F)$ -homotopic rel.end-points. Let $\gamma_1, \dots, \gamma_n$ be a sequence of $f(F)$ -chains realizing $f(F)$ -homotopy from $f(\alpha)$ to $f(\beta)$. Choose an E_b -lift λ_i of γ_i for each $1 < i < n$ and put $\lambda_1 = \alpha$, $\lambda_n = \beta$. To show λ_i is E -homotopic to λ_{i+1} it suffices to consider the case γ_{i+1} is obtained from γ_i via an $f(F)$ -expansion. Let Γ be the chain obtained from λ_{i+1} by dropping the expansion vertex. Notice Γ is an E_b^2 -chain and $f(\Gamma) = f(\lambda_i)$. Hence Γ is E_a -close to λ_i and it is E_a^2 -homotopic rel.end-points to λ_i . Since λ_{i+1} is an E_b -expansion of Γ , it is E -homotopic to λ_i .

We still need to show that the endpoints of α and β are sufficiently close. Pick an entourage $F_1 \subset F$ of X so that any $f(F_1)$ -chain in Y lifts to an F -chain in X . Assume α and β are F_1 -chains starting from x_0 such that $f(\alpha)$ and $f(\beta)$ are $f(F_1)$ -homotopic. Extend $f(\alpha)$ to μ by adding the end-point of $f(\beta)$ and lift μ to an F -chain α' . Now $f(\alpha')$ and $f(\beta)$ are $f(F)$ -homotopic, so by the previous case α' is E -homotopic to β rel.end-points. Since α' with end-point removed is E_a -close to α , we get α is E^2 -homotopic to β . \square

5.5 says the difference between 2.10 and Conditions C1-2 of 5.1 is that for uniform covering maps one has existence and uniqueness of lifts of chains (assuming the

chains are sufficiently fine - that comes from existence of an entourage transverse to the covering map) and for generalized uniform covering maps one has existence and approximate uniqueness of lifts of chains.

Let us show that Condition GP2 is superfluous and Condition GP1 in 5.1 follows from C1 and C2 provided the fibers of f are complete.

Proposition 5.6. *If $f: X \rightarrow Y$ satisfies Conditions GP1 and C1-2 of 5.1, then it satisfies Condition GP2 of 5.1.*

Proof. Suppose any two F -chains α and β originating from the same point must be E -homotopic if $f(\alpha)$ and $f(\beta)$ are F_1 -homotopic, where $F_1 = f(F)$. Given $(\tilde{f}(d_1), \tilde{f}(d_2)) \in F_1^*$, where $d_1, d_2 \in GP(X, x_0)$, we may assume F -terms of both d_1 and d_2 are realized by F -chains α and β , respectively. Since $f(\alpha)$ is F_1 -homotopic to $f(\beta)$, α is E -homotopic to β resulting in $(d_1, d_2) \in E^*$. \square

Lemma 5.7. *Suppose $f: X \rightarrow Y$ is a uniformly continuous map with complete fibers. f is a generalized uniform covering map if it generates the uniform structure on Y and the Conditions C1-2 of 5.1 are satisfied.*

Proof. The only task is to show that any generalized path $c = \{[c_E]\}$ in Y starting at $f(x_0)$ has a lift starting at x_0 . Given an entourage E of X , choose an entourage $\alpha(E) \subset E$ so that $f\alpha(E)$ -chains in Y lift to E -chains. Let $\widetilde{c_{f\alpha(E)}}$ be an E -lift of $c_{f\alpha(E)}$ and define x_E to be the endpoint of $\widetilde{c_{f\alpha(E)}}$. To see that $\{x_E\}$ is Cauchy let E be an entourage of X and choose an entourage H of X so that two H -chains are E -homotopic if their images are $f(H)$ -homotopic. Suppose $F_1, F_2 \subset H$. Then $\widetilde{c_{f\alpha(F_1)}}$ and $\widetilde{c_{f\alpha(F_2)}}$ are H -chains with images $c_{f\alpha(F_1)}$ and $c_{f\alpha(F_2)}$ respectively. Since $f\alpha(F_1), f\alpha(F_2) \subset f(H)$, $c_{f\alpha(F_1)}$ and $c_{f\alpha(F_2)}$ are $f(H)$ -homotopic so $(x_{F_1}, x_{F_2}) \in E$. Let x be a limit point of $\{x_E\}$. We will extend lifts of chains c_E to x to form a generalized path $d = \{[d_E]\}$ so $f(d) = c$.

Given an entourage E of X choose an entourage $\beta(E) \subset E$ so that two $\beta(E)$ -chains are E -homotopic if their images are $f\beta(E)$ -homotopic and choose an entourage $\gamma(E) \subset \beta(E)$ so that for any entourage $F \subset \gamma(E)$, $(x_F, x) \in \beta(E)$. Given an entourage E of X define d_E to be $\widetilde{c_{f\alpha\gamma(E)}}$ extended to x . Since $\widetilde{c_{f\alpha\gamma(E)}}$ and d_E are both $\beta(E)$ -chains with $f\beta(E)$ -homotopic images, $f(d_E) = c_{f\alpha\gamma(E)}$ which is E -homotopic to $c_{f(E)}$ so $f(d) = c$. To see that d is in fact a generalized path suppose $F \subset E$ are entourages of X and consider the entourage $H = f\alpha(\alpha\gamma(F) \cap \alpha\gamma(E))$. Choose an $\alpha\gamma(F) \cap \alpha\gamma(E)$ -lift h of c_H and notice it is a $\beta(F)$ -chain whose image is $f\beta(F)$ -homotopic to $c_{f\alpha\gamma(F)}$. Therefore h is F -homotopic to d_F . Similarly h is E -homotopic to d_E so we have d_F E -homotopic to d_E . \square

5.1. Generalized uniform covering maps and uniformly joinable spaces.

Theorem 5.8. *Suppose X is uniformly joinable chain connected Hausdorff uniform space. If $f: X \rightarrow Y$ generates the uniform structure of Y , then the following conditions are equivalent:*

- a. f is a generalized uniform covering map.
- b. f satisfies Conditions GP1-2.
- c. $\tilde{f}: GP(X, x_0) \rightarrow GP(Y, f(x_0))$ is a uniform equivalence for some $x_0 \in X$.

Proof. The equivalence of b) and c) follows from 5.2. Suppose f satisfies Conditions GP1-2.

Proof of C1. Given an entourage E of X choose an entourage F_1 of X so that two generalized paths starting at the same point are E -homotopic provided their images are $f(F_1)$ -homotopic. Choose an entourage F of Y so that any $(x, y) \in F$ can be joined by a generalized path that is $f(F_1)$ -short. Suppose $(f(x), y) \in F$. Join $f(x)$ and y by a generalized path c that is $f(F_1)$ -short. Now c lifts to a generalized path \tilde{c} starting at x . Let y' be the endpoint of \tilde{c} . Since c is $f(F_1)$ -homotopic to the constant path at $f(x)$, \tilde{c} is E -homotopic to the constant path at x . In particular $(x, y') \in E$.

Proof of C2. Given an entourage E of X choose an entourage G of Y so that $(\tilde{f}(c), \tilde{f}(d)) \in G^*$ implies $(c, d) \in E^*$ for any two generalized paths c and d originating from the same point. Choose an entourage $F \subset H = E \cap f^{-1}(G)$ of X such that any pair $(x, y) \in F$ can be joined by a generalized H -short path $c(x, y)$. Given an F -chain α create a generalized uniform path $p(\alpha)$ by replacing each of its edges $[x_i, x_{i+1}]$ with $c(x_i, x_{i+1})$. Suppose $f(\alpha)$ is $f(F)$ -homotopic to $f(\beta)$. In that case $(\tilde{f}(p(\alpha)), \tilde{f}(p(\beta))) \in G^*$. Therefore $(p(\alpha), p(\beta)) \in E^*$. As $p(\alpha)_E = [\alpha]$ and $p(\beta)_E = [\beta]$, α is E -homotopic to β . \square

Corollary 5.9. *Let X_1, X_2, X_3 be uniformly joinable chain connected Hausdorff uniform spaces. Suppose $f: X_1 \rightarrow X_2$ and $g: X_2 \rightarrow X_3$ generate the uniform structure of their ranges. If any two of $f, g, h = g \circ f$ are generalized uniform covering maps, then so is the third.*

Theorem 5.10. *The projection $\pi_X: GP(X, x_0) \rightarrow X$ is a universal generalized uniform covering map in the category of uniformly joinable chain connected Hausdorff spaces.*

Proof. Suppose X is chain connected and uniformly joinable. By 4.15 and 5.8 π_X is a generalized uniform covering map. If $f: X \rightarrow Y$ is a generalized uniform covering map, then for any $x_0 \in X$ the induced map $\tilde{f}: GP(X, x_0) \rightarrow GP(Y, f(x_0))$ is a uniform equivalence, so we put $g = \pi_Y \circ \tilde{f}^{-1}: GP(Y, f(x_0)) \rightarrow X$. It is clearly a lift of π_Y . Since \tilde{f} is a uniform equivalence and $GP(Y, f(x_0))$ is joinable, g is a generalized uniform covering map by 5.8. \square

Corollary 5.11. *If X is a uniform Poincare space, then the natural function from \tilde{X} to $GP(X, x_0)$ is a uniform equivalence.*

Proof. Use 5.10 to produce a lift $\alpha: GP(X, x_0) \rightarrow \tilde{X}$ of the projection $GP(X, x_0) \rightarrow X$ that generates the uniform structure of X . That lift is the inverse of $\beta: \tilde{X} \rightarrow GP(X, x_0)$ (β sends a path in X to the induced generalized uniform path). Indeed, we can apply 5.3 to conclude both $\alpha \circ \beta$ and $\beta \circ \alpha$ are identities. \square

Theorem 5.12. *Suppose $f: X \rightarrow Y$ is a generalized uniform covering map and $y_0 = f(x_0)$. If Z is uniformly joinable chain connected, then the following are equivalent for any $z_0 \in Z$ and any uniformly continuous $g: Z \rightarrow Y$ so that $g(z_0) = y_0$:*

- a. *There is a uniformly continuous lift $h: Z \rightarrow X$ of g satisfying $h(z_0) = x_0$,*
- b. *The image of $\tilde{\pi}_1(g): \tilde{\pi}_1(Z, z_0) \rightarrow \tilde{\pi}_1(Y, y_0)$ is contained in the image of $\tilde{\pi}_1(f): \tilde{\pi}_1(X, x_0) \rightarrow \tilde{\pi}_1(Y, y_0)$*

Moreover, if g is a generalized uniform covering map and has a uniformly continuous lift h , then h is a generalized uniform covering map provided X is joinable.

Proof. b) \implies a). Given $z \in Z$ pick $c \in GP(Z, z_0)$ from z_0 to z and let $d \in GP(X, x_0)$ satisfy $\tilde{f}(d) = \tilde{g}(c)$. That d is unique (once c is chosen) and its end-point is our choice for $h(z)$. If c' is another generalized path from z_0 to z with the resulting $d' \in GP(X, x_0)$, then $\tilde{g}(c * (c')^{-1})$ is a generalized loop in Y at y_0 and we can choose a generalized loop $e \in GP(X, x_0)$ so that $\tilde{f}(e) = \tilde{g}(c * (c')^{-1})$. Now $\tilde{f}(e * d') = \tilde{f}(e) * \tilde{g}(c') = \tilde{g}(c) = \tilde{f}(d)$, so $e * d' = d$ and the end-points of d' and d are the same. Hence $h(z)$ is independent on the choice of generalized path c .

It remains to show h is uniformly continuous and here is where we use Conditions GP1-2. Given an entourage E of X pick an entourage F of Y so that any F -short generalized path in Y lifts to an E -short generalized path in X . Next choose an entourage G of Z satisfying $g(G) \subset F$. Finally, choose an entourage H of Z such that any two points $(z, z') \in H$ can be connected by a G -short generalized path. Pick $c \in GP(Z, z_0)$ from z_0 to z and then a G -short c' from z to z' . The difference between $\tilde{h}(c)$ and $\tilde{h}(c * c')$ is F -short, so they have lifts to X that differ by an E -short path. The conclusion is that $(h(z), h(z')) \in E$ which means $h(H) \subset E$, i.e. h is uniformly continuous.

Assume X is joinable, g is a generalized uniform covering map and has a uniformly continuous lift h . In view of 5.4 it suffices to show h generates the uniform structure of its range. Since $\tilde{g} = \tilde{h} \circ \tilde{f}$, \tilde{g} is a uniform equivalence and $GP(X, x_0)$ is uniformly joinable. \square

Corollary 5.13. *Let X be a uniformly joinable and chain connected space. The projection $\pi_X: GP(X, x_0) \rightarrow X$ is a uniform covering map if and only if there is an entourage E of X such that the natural homomorphism $\tilde{\pi}_1(X, x_0) \rightarrow \pi_1(R(X, E), x_0)$ is a monomorphism.*

Proof. Suppose $\pi_X: GP(X, x_0) \rightarrow X$ is a uniform covering map and choose an entourage E of X such that E^* is transverse to π_X . That means $(c, d) \in E^*$ implies $c = d$ if c and d are generalized paths with the same end-point. Suppose c and d are generalized loops at x_0 so that $c_E = d_E$. That implies $c^{-1} * d$ is E -short, hence $c^{-1} * d$ is trivial and $c = d$.

Suppose the natural homomorphism $\tilde{\pi}_1(X, x_0) \rightarrow \pi_1(R(X, E), x_0)$ is a monomorphism. If c and d are two generalized paths from x_0 to x such that $b = c^{-1} * d$ is E -short, then b_E is trivial and b must be trivial. That means $c = d$ and π_X is a uniform covering map. \square

Theorem 5.14. *Suppose X has a countable base of entourages and is uniformly joinable. If $f: X \rightarrow Y$ is a generalized uniform covering map, then the fibers of f are complete.*

Proof. Choose a base $\{E_n\}_{n=1}^\infty$ of entourages of X satisfying the following conditions:

- (1) Any pair $(x, y) \in E_{n+1}$ admits a generalized uniform path $c_n(x, y)$ from x to y whose E_n -term is the edge-path $e(x, y)$.
- (2) If α and β are two E_{m+1} -chains originating at the same point, then they are E_m -homotopic if $f(\alpha)$ is $f(E_{m+1})$ -homotopic to $f(\beta)$.

Given a Cauchy sequence in a fiber $f^{-1}(y)$ of f we may choose its subsequence $\{x_n\}_{n=1}^\infty$ such that $(x_k, x_m) \in E_{n+1}$ for $k, m \geq n$.

Let α_1 be the edge-path $e(x_1, x_2)$. Given an E_{n+1} -chain α_n from x_1 to x_{n+1} construct α_{n+1} by replacing each link $e(u, v)$ of α_n by the E_{n+2} -term of $c_n(u, v)$ and

then concatenating all of it with $e(x_{n+1}, x_{n+2})$. Notice $\{f(\alpha_n)\}_{n=1}$ is a generalized uniform loop at y , so it has a lift $\{\beta_n\}$ from x_1 to some $x \in f^{-1}(y)$. If x is not the limit of $\{x_n\}_{n=1}$, then there is $m \geq 1$ with no x_i belonging to $B(x, E_m)$. As $f(\alpha_{m+1})$ is $f(E_{m+1})$ -homotopic to $f(\beta_{m+1})$, α_{m+1} is E_m -homotopic to β_{m+1} . In particular, their end-points are E_m -close. Thus $(x, x_{m+1}) \in E_m$, a contradiction. \square

6. GENERALIZED PATHS RELATIVE TO SPACES

In this section we expand an idea of Krasinkiewicz and Minc [21] to define generalized uniform paths of X via an embedding in a uniform space T with nice local properties. We require T to be uniformly locally path-connected and the embedding $X \rightarrow T$ satisfies the following analog of uniform semi-local simply connectedness: Given an entourage E of T there is an entourage $F \subset E$ of T such that any loop in $B(x, F)$ is contractible in $B(X, E)$ for all $x \in X$ (here $B(X, E)$ is $\bigcup_{x \in X} B(x, E)$).

One important case is that of T being uniformly locally simply-connected as every uniform Hausdorff space X embeds in the Tychonoff cube I^J for some J (that embedding is simply via the set of all uniformly continuous functions $X \rightarrow I$, so that is what one can choose for index set J).

Another important case is of $T = X$ and X being a uniform Poincare space.

From now on we assume X is chain-connected. In this case one can define generalized paths following [21] (only the compact metric case is discussed there): $GP_T(X, x_0)$ is the set of **generalized paths** in T from x_0 to points of X . A **generalized path** is a family $\{[c_E]\}_{E \in \mathcal{E}}$ of homotopy classes of paths c_E in $B(X, E)$ with common end-point $x \in X$ such that for $F \subset E$ the path c_F is homotopic to c_E in $B(X, E)$ rel. end-points. Given an entourage F of T we define an entourage F_* of $GP_T(X, x_0)$ as the set of pairs $(\{[c_E]\}_{E \in \mathcal{E}}, \{[d_E]\}_{E \in \mathcal{E}})$ such that $c_F^{-1} * d_F$ is homotopic rel. end-points to a path contained in $B(z, F)$ for some $z \in X$.

Notice if $T = X$ and X is a uniform Poincare space, $GP_X(X, x_0)$ is simply \tilde{X} .

Our goal is to discuss the connection between $GP(X, x_0)$ and $GP_T(X, x_0)$.

Given an entourage E of T let $u(E) \subset E$ be an entourage of T such that any loop in $B(x, u(E)^2)$ is contractible in $B(X, E)$ for all $x \in X$. Let $v(E) \subset E$ be an entourage of T such that any two points in $B(x, v(E))$ can be connected by a path in $B(x, E)$ for all $x \in T$. Put $w(E) = v(u(E))$.

Given a $w(E)$ -chain $c = x_0, \dots, x_k$ in X from x_0 to x choose a path α_m from x_m to x_{m+1} in $B(x_m, u(E))$. Observe that the homotopy type of α_m (rel. end-points) in $B(X, E)$ does not depend on the choice of α_m . Therefore one has a well-defined path-homotopy class $i(c)$ from x_0 to x in $B(X, E)$.

Lemma 6.1. *If c is homotopic to d rel. end-points in $R(X, w(E))$, then $i(c) = i(d)$.*

Proof. It suffices to consider two cases: reduction of a vertex of x_0, \dots, x_k or expansion of x_0, \dots, x_k by a vertex.

If a vertex x_{m+1} is dropped from x_0, \dots, x_k , then the concatenation of paths α_m and α_{m+1} is replaced by a path β straight from x_m to x_{m+2} . Since $\alpha_m * \alpha_{m+1} * \beta^{-1}$ is a loop in $B(x_m, u(E)^2)$, it is null-homotopic in $B(X, E)$ and $\alpha_0 * \dots * \alpha_{k-1}$ is homotopic rel. end-points to the concatenation in which $\alpha_m * \alpha_{m+1}$ is replaced by β .

The case of expansion of x_0, \dots, x_k by one vertex is essentially covered by the first case. \square

Given an entourage F of T and given a path α from x_0 to $x \in X$ in $B(X, F)$ construct the homotopy class $j(\alpha)$ of a path from x_0 to x in $R(X, F^6)$ as follows:

For each $t \in [0, 1]$ find $x(t) \in X$ so that $(\alpha(t), x(t)) \in F$ (obviously, we want $x(0) = x_0$ and $x(1) = x$). Then find a subdivision $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1$ of the unit interval I such that $\alpha[t_m, t_{m+1}]$ is contained in $B(z, F^2)$ for some $z \in X$. We need to take F^2 since $B(z, F) \subset \text{Int}B(z, F^2)$. Let $j(\alpha)$ be the homotopy class of the F^6 -chain $x(0), \dots, x(t_k)$ in $R(X, F^6)$.

Lemma 6.2. *$j(\alpha)$ does not depend on the choice of points $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1$ and $j(\alpha) = j(\beta)$ in $R(X, F^{12})$ if α is homotopic to β in $B(X, F)$ rel. end-points.*

Proof. To show independence of $j(\alpha)$ of the choice of points $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1$ it suffices to consider the case of expanding $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1$ by adding extra s , $t_m \leq s \leq t_{m+1}$. The reason is that any two subdivisions of the unit interval can be combined by adding one point at the time. Since $(x(t_m), x(s)) \in F^6$ and $(x(t_{m+1}), x(s)) \in F^6$, the chain $x(0), \dots, x(t_m), x(s), x(t_{m+1}), \dots, x(t_k)$ is an F^6 -expansion of $x(0), \dots, x(t_k)$ and is homotopic to $x(0), \dots, x(t_k)$ rel. end-points in $R(X, F^6)$.

Suppose $H: I \times I \rightarrow B(X, F)$ is a homotopy rel. end-points from α to β . There is an equally spaced subdivision $0 = t_0 \leq t_1 \leq \dots \leq t_k = 1$ of the unit interval I so that $H([t_m, t_{m+1}] \times [t_n, t_{n+1}]) \subset B(z, F^2)$ for some $z \in X$. To conclude $j(\alpha) = j(\beta)$ in $R(X, F^{12})$ it suffices to apply the following:

Observation. If E is an entourage of X and $x_0, \dots, x_k, y_0, \dots, y_k$ are two E -chains joining x_0 to x , then they are homotopic in $R(X, E^2)$ rel. end-points if $(x_n, y_n) \in E$ for all $n \leq k$.

Proof of Observation. Create an E^2 -chain $x_0, y_0, \dots, x_k, y_k$ and notice it can be reduced to both x_0, \dots, x_k and y_0, \dots, y_k in $R(X, E^2)$. \square

Lemma 6.3. *Let E be an entourage of T and F be an entourage with $F^{12} \subset w(E)$. If α is a path in $B(X, F)$ then $i(j(\alpha))$ is homotopic to α in $B(X, E)$. Similarly, let E be an entourage of T and F be an entourage with $F^{12} \subset E$. If γ is a path in $R(X, w(F))$, then $j(i(\gamma))$ is homotopic to γ in $R(X, E)$.*

Proof. Say $j(\alpha)$ is the homotopy class of $x_0 = x(t_0), \dots, x(t_k)$. For each $i < k$, $\alpha(t_i), \alpha(t_{i+1}), x(t_{i+1}) \in B(x(t_i), w(E))$ so there are paths from $x(t_i)$ to $\alpha(t_i)$ and from $x(t_{i+1})$ to $\alpha(t_{i+1})$ that are contained in $B(x(t_i), u(E))$. Therefore $i(j(\alpha))$ is homotopic to α in $B(X, E)$. Now suppose γ is represented by the $w(E)$ -chain x_0, \dots, x_k . Notice $j(i(\gamma))$ is the same chain in $R(X, E)$ since for each i , α_i (from the definition of j) is contained in $B(x_i, u(F)) \subset B(x_i, F^2)$. \square

Now we are in a position to define $i: GP(X, x_0) \rightarrow GP_T(X, x_0)$ and $j: GP_T(X, x_0) \rightarrow GP(X, x_0)$.

Given $c = \{[c_F]\}_{F \in \mathcal{E}} \in GP(X, x_0)$ (from x_0 to x) assume each c_F is realized by an F -chain $x_0, \dots, x_{k(F)}$ in $R(X, F)$. Given an entourage E of T use 6.1 to notice $i(c_F)$ is independent of the choice of $F \subset w(E)$. By putting $i(c)_E = i(c_F)$ we get a well-defined element of $GP_T(X, x_0)$. Similarly, given an element $\alpha = \{[\alpha_F]\}_{F \in \mathcal{E}} \in GP_T(X, x_0)$ use 6.2 to notice that for any entourage E of T the element $j(\alpha_F)$ does not depend on F provided $F^{12} \subset E$. Thus, putting $j(\alpha) = \{j(\alpha_F)\}_{E \in \mathcal{E}}$ we get a well-defined element of $GP(X, x_0)$.

Theorem 6.4. *$i: GP(X, x_0) \rightarrow GP_T(X, x_0)$ and $j: GP_T(X, x_0) \rightarrow GP(X, x_0)$ are uniformly continuous and inverse to each other.*

Proof. For uniform continuity of i let us show that $i(w(E)^*) \subset E_*$. Let $(c, d) \in w(E)^*$. Then $c_{w(E)}^{-1} * d_{w(E)}$ is homotopic to $e(x, y)$ in $R(X, w(E))$ where x and y are the endpoints of c and d respectively. Then we have $i(c)_E^{-1} * i(d)_E = i(c_{w(E)})^{-1} * i(d_{w(E)}) = i(c_{w(E)}^{-1} * d_{w(E)}) = i(e(x, y))$ so $(i(c), i(d)) \in E_*$. Similarly, for continuity of j , we show $j(F_*) \subset E^*$ for $F^{12} \subset E$. Let $(\alpha, \beta) \in F_*$ and x and y be the endpoints of α and β respectively. Then $\alpha_F^{-1} * \beta_F$ is homotopic in $B(X, F)$ to some path γ from x to y that is contained in $B(z, F)$ for some $z \in X$. Then we have $j(\alpha)_E^{-1} * j(\beta)_E = j(\alpha_F)^{-1} * j(\beta_F) = j(\alpha_F^{-1} * \beta_F) = j(\gamma)$ in $R(X, E)$. Notice that in $R(X, E)$, $j(\gamma)$ is homotopic to $e(x, y)$.

Let $\alpha \in GP_T(X, x_0)$ and consider $i(j(\alpha))$. We have $i(j(\alpha))_E = i(j(\alpha)_{w(E)}) = i(j(\alpha_F))$ where $F^{12} \subset w(E)$. By 6.3, $i(j(\alpha_F))$ is homotopic to α_E in $B(X, E)$. Now let $c \in GP(X, x_0)$ and consider $j(i(c))$. We have $j(i(c))_E = j(i(c)_F) = j(i(c_{w(F)}))$ for $F^{12} \subset E$. Again, by 6.3, $j(i(c_{w(F)}))$ is homotopic in $R(X, E)$ to c_E . \square

Corollary 6.5. *If X is a metric continuum, then $\tilde{\pi}_1(X, x_0)$ is isomorphic to the first shape group of (X, x_0) .*

Proof. Embed X in the Hilbert cube Q . As in the proof of 6.4, $\tilde{\pi}_1(X, x_0)$ is isomorphic (also in the category of topological groups) to the group of generalized loops of X in Q at x_0 . That is the same as the inverse limit of $\{\pi_1(U_n)\}$ (each with the discrete topology), where U_n is the $\frac{1}{n}$ -ball of X in Q , and that is exactly the first shape group of (X, x_0) (see [9] or [24]). \square

Definition 6.6 (cf. [9, p.88]). A pointed continuum (X, x_0) is called **pointed 1-movable** if $\text{pro-}\pi_1(X, x_0)$ satisfies the Mittag-Leffler condition.

Corollary 6.7. *For a metric continuum X the following conditions are equivalent:*

- a. X is joinable,
- b. X is pointed 1-movable.
- c. X is uniformly joinable.

Proof. a) \implies b). Embed X in the Hilbert cube Q . As in the proof of 6.4 joinability of X is equivalent to the property that every two points $x, y \in X$ there is a sequence of paths a_n joining x to y in the $(1/n)$ -neighborhood U_n of X such that a_{n+1} is homotopic to a_n rel. end-points in U_n . That coincides with the original definition of joinability of X given by Krasinkiewicz and Minc [21]. The main result of [21] states that joinable continua have the fundamental pro-group satisfying the Mittag-Leffler condition.

b) \implies c). By [9, Theorem 6.1.7], we have $\lim^1(\text{pro-}\pi_1(X, x_0)) = 0$. By 4.12 (or [21]), X is joinable. Applying 4.9 and 4.11 gives X being uniformly joinable. c) \implies a). Follows from 4.5 since any metric continuum is chain connected. \square

Since all subcontinua of surfaces are pointed 1-movable (see [9, Theorem 7.1.7]), one has the following:

Corollary 6.8. *All subcontinua of surfaces are uniformly joinable (that includes the suspension of the Cantor set and the Hawaiian Earring). The dyadic solenoid is not joinable.*

In connection to 6.8 let us point out the boundary of any word-hyperbolic group is compact and metrizable [17] and the boundary of any one-ended word-hyperbolic group is locally connected [5] (hence pointed 1-movable). Also, pointed 1-movability is related to semi-stability at infinity of groups (see [25] and [15]).

Corollary 6.9. *If X is a uniformly joinable metric continuum, then the following conditions are equivalent:*

- a. *The projection $\pi_X : GP(X, x_0) \rightarrow X$ is a uniform covering map,*
- b. *$\tilde{\pi}_1(X, x_0)$ is countable,*
- c. *$\tilde{\pi}_1(X, x_0)$ is finitely generated.*

Proof. Embed X in the Hilbert cube Q . We show that $\pi_X : GP(X, x_0) \rightarrow X$ is a uniform covering map if and only if there is a closed neighborhood N of X in Q with $\tilde{\pi}_1(X, x_0) \rightarrow \pi_1(N, x_0)$ a monomorphism and N the homotopy type of a compact polyhedron. That condition is known to be equivalent to b) and c) (see [13] or [24, Corollary 8 on p.177]). According to 5.13 $\pi_X : GP(X, x_0) \rightarrow X$ is a uniform covering map if and only if there is an entourage E of X so that $\tilde{\pi}_1(X, x_0) \rightarrow \pi_1(R(X, E), x_0)$ is a monomorphism. Suppose such an E exists and let F be an entourage of X with $F^{12} \subset E$. Then $\tilde{\pi}_1(X, x_0) \rightarrow \pi_1(R(X, w(F)), x_0) \rightarrow \pi_1(B(X, F), x_0) \rightarrow \pi_1(R(X, E), x_0)$ is a monomorphism (see 6.3) so $\tilde{\pi}_1(X, x_0) \rightarrow \pi_1(B(X, F), x_0)$ is as well. Note that there is a closed neighborhood N of X in Q with $N \subset B(X, F)$ and N the homotopy type of a compact polyhedron. Similarly, if such an N exists, find $\varepsilon > 0$ so that $B(X, E_\varepsilon) \subset N$ where $E_\varepsilon = \{(x, y) \in Q \times Q : d(x, y) < \varepsilon\}$. Take an entourage F such that $F^{12} \subset w(E_\varepsilon)$. Then $\tilde{\pi}_1(X, x_0) \rightarrow \pi_1(B(X, F), x_0) \rightarrow \pi_1(R(X, w(E_\varepsilon)), x_0) \rightarrow \pi_1(B(X, E_\varepsilon), x_0)$ is a monomorphism so $\tilde{\pi}_1(X, x_0) \rightarrow \pi_1(R(X, w(E_\varepsilon)), x_0)$ is as well. \square

Theorem 6.10. *Suppose X is a path-connected uniform space. If the projection $\pi : \tilde{X} \rightarrow X$ is a generalized uniform covering map, then X is a uniform Poincare space and π is a uniform covering map.*

Proof. X is uniformly locally path-connected by 2.12. It suffices to show X is uniformly semi-locally simply connected. Suppose for each entourage E of X there is a point x_E and a loop α_E at x_E in $B(x, E)$ that is non-trivial in X . Pick a path γ_E from x_0 to x_E . By picking points on the loop $\beta_E = \gamma_E * \alpha_E * \gamma_E^{-1}$ that belong to the image of γ_E only one can define an E^* -chain in \tilde{X} starting from the trivial loop at x_0 and ending at β_E . The same chain works for $\omega_E = \gamma_E * \gamma_E^{-1}$ but this time we do not go around α_E . Thus we have two E^* -chains in \tilde{X} with the same projection in X , so they should be $X \times X$ -homotopic for some E , a contradiction. \square

7. COMPARISON TO BERESTOVSKII-PLAUT UNIFORM COVERS

Berestovskii and Plaut used an analogue of the the Schreier construction for topological groups [1] to create an inverse limit construction [3] for a uniform space X . We recall their construction (which we denote by \tilde{X}_{BP} as \tilde{X} is used by us for classical universal cover) below, and compare their inverse limit space \tilde{X}_{BP} to $GP(X, x_0)$.

Let X be an uniform space with a fixed base point x_0 . For any entourage E an E -chain starting at x_0 and ending at $x \in X$ is a finite sequence of points $\{x_0, \dots, x_n = x\}$ such that $(x_i, x_{i+1}) \in E$ for all $0 \leq i \leq n-1$. An E -extension of a E -chain $\{x_0, \dots, x_n = x\}$ is a E -chain $\{x_0, \dots, x_i, y, x_{i+1}, \dots, x_n = x\}$, with $0 \leq i < n$. An E -homotopy is a finite sequence of E -extensions (or their obvious analogues E -contractions). X_E is the set of all E -homotopy classes $[c]_E$ of E -chains c . For any entourage $F \subset E$ define \hat{F} as follows: $([c]_E, [d]_E) \in \hat{F}$ if $([c]_E, [d]_E) =$

$([x_0, \dots, x_n, y]_E, [x_0, \dots, x_n, z]_E)$ with $(y, z) \in F$. The collection of all such \hat{F} is a base for the uniformity on X_E . If $F \subset E$ is an entourage, there is a natural map $\phi_{EF} : X_F \rightarrow X_E$ which sends $[c]_F$ to $[c]_E$ and generates the uniform structure of X_E . With hindsight one may say the structure on X_E mimicks the basic topology on \tilde{X} .

The inverse limit \tilde{X}_{BP} of $\{X_E\}_{E \in \mathcal{E}}$ is given the inverse limit uniformity. Thus \tilde{X}_{BP} is equivalent to our space $GP(X, x_0)$. The advantage of our description is a closer connection to the classical universal cover \tilde{X} and generalized paths of Krasinkiewicz-Minc.

For the same reason the deck group $\delta_1(X)$ of [3] is isomorphic to our fundamental uniform group $\tilde{\pi}_1(X, x_0)$. Again, the advantage of our approach is the connection between $\tilde{\pi}_1(X, x_0)$ and the fundamental shape group in case of metrizable compact spaces X .

The basic class of uniform spaces for which the approach in [3] works is the class of coverable spaces. A uniform space X is **coverable** if there is a uniformity base of entourages E (including $X \times X$) such that the projections $\tilde{X}_{BP} \rightarrow X_E$ are surjective. In our language that means for every path α in $R(X, E)$ there is a generalized path $c = \{c_F\}_{F \in \mathcal{E}}$ such that c_E is homotopic rel.end-points to α . Thus every coverable space is uniformly joinable and our theory of generalized uniform covering maps induces most basic results of [3]. A natural question arises:

Problem 7.1. *Is every uniformly joinable chain-connected space coverable?*

The relevance of 7.1 is that it would imply a positive answer to Problem 106 of [3] for chain-connected spaces (that problem asks if X is coverable provided $\tilde{X}_{BP} \rightarrow X$ is a uniform equivalence). Indeed, 4.6 implies X is uniformly joinable if $\tilde{X}_{BP} \rightarrow X$ is a uniform equivalence.

There are two obvious strategies to solve 7.1 positively:

- Given an entourage E of X choose an entourage $F \subset E$ with the property that any pair $(x, y) \in F$ can be connected by a generalized path $c(x, y)$ so that its E -term is the edge $e(x, y)$. Try to show $\tilde{X}_{BP} \rightarrow X_F$ is surjective.
- Given an entourage E of X define $G(E)$ as all pairs $(x, y) \in E$ with the property that there are generalized paths c from x_0 to x and d from x_0 to y such that $(c^{-1} * d)_E$ is homotopic in $R(X, E)$ to the edge $e(x, y)$ (as X is uniformly joinable $G(E)$ contains F above and is an entourage). Try to show $\tilde{X}_{BP} \rightarrow X_{G(E)}$ is surjective.

Notice Strategy b) is a natural reaction once one realizes Strategy a) fails.

Let us show two examples negating the above strategies.

Example 7.2. Consider a regular hexagon with one edge ab of size 1 removed. Let E be pairs of distance at most 3 and F are pairs of distance at most 1.

Proof. To check that any F -short pair can be connected by the right path (notice there is only one path for every pair anyhow) it suffices to prove it for (a, b) . Let α be the genuine path in X from a to b . We can eliminate first all non-vertex points, then all vertices and α_E is homotopic in $R(X, E)$ to $e(a, b)$. Here is the problem: consider chain $x_0 = a, x_1 = b$ in F and suppose there is a generalized path α whose F -term is homotopic to $\{x_0, x_1\}$. There is no way a 1-chain from a to b to be 1-homotopic to $\{x_0, x_1\}$ (consider the last point removed prior to arriving

at pair $\{x_0, x_1\}$) and such generalized path would produce a chain of that kind. \square

Example 7.3. Consider a regular hexagon with one edge ab of size 1 removed. Add the center c of the hexagon plus a vertical regular hexagon with bottom ac that we remove. The resulting X and $E = \{(x, y) | \text{dist}(x, y) \leq 1 = \text{dist}(a, b)\}$ have the property that $(a, b) \in G(E)$ but they cannot be joined by a generalized path in X whose $G(E)$ -term is the edge as $(p, c) \notin G(E)$ for any point p belonging to the first hexagon.

Example 7.2 says there is an error in [29]. Indeed, in the proof of Proposition 5 one considers the entourage F^* in X_E consisting of pairs of homotopy classes of paths (a, b) such that their end-points x and y satisfy $(x, y) \in F$, $a^{-1} * b$ is homotopic rel.end-points to the edge $e(x, y)$, and there are generalized paths c and d so that $c_E = a$ and $d_E = b$. The entourage G in $\tilde{X} = \tilde{X}_{BP}$ is defined as pairs (c, d) so that $(c_E, d_E) \in F^*$ and Proposition 5 claims the projection $\tilde{X}_G \rightarrow \tilde{X}$ is a homeomorphism for all such G . Once that holds the proof of Lemma 6 in [29] gives that $\tilde{X} \rightarrow X_{\pi(G)}$ is surjective provided all such defined entourages G form a base of entourages of \tilde{X} which is so if X is uniformly joinable. However, $\pi(G)$ (π being the projection from \tilde{X} to X) is exactly F and Example 7.2 shows the projection $\tilde{X} \rightarrow X_{\pi(G)}$ may not be surjective.

The best way to explain to a topologist the philosophical difference between Berestovskii-Plaut notion of coverability and our notion of uniform joinability is to point out the latter is a UV -type condition and the former one is the same condition replaced by existence of a base where V can be chosen equal to U . In Siebenmann's thesis he starts from UV -type conditions and produces an end of a manifold. Such an end can be intuitively explained by requiring $V = U$ for some base of neighborhoods U of infinity and some UV -type condition. That means an answer to 7.1 could be positive but a topologist would be sceptical without adding extra conditions on the space X .

From algebraic point of view uniform joinability corresponds to the Mittag-Leffler condition and, for inverse sequences of groups, Mittag-Leffler condition is indeed equivalent to existence of an inverse sequence of epimorphisms. That analogy may lead to a larger dose of optimism in a positive answer to 7.1. However, one may point out that Theorem 7 of [2] characterizes coverability of a locally compact topological group G as being equivalent to G being connected and locally arcwise connected. Thus, 7.1 has a positive answer for locally compact topological groups which may be analogous to the Mittag-Leffler condition for inverse sequences of groups.

Summing up: uniform joinability is of a shape-theoretical nature and coverability is more of a geometrical nature.

In [3] (on p.1751, the paragraph below Theorem 3) the authors mention they do not know whether the composition of pro-discrete covers between coverable spaces (or uniform spaces in general) is a pro-discrete cover but in the case of topological groups it is so. Let us point out an example resolving that question in the negative even for discrete covers.

Example 7.4. Consider the case of subgroups $G_1 \subset G_2$ of a group G_3 such that G_2 is normal in G_3 , G_1 is normal in G_2 but G_1 is not normal in G_3 . In case of $G_1 = \mathbb{Z}$, the group of integers, $G_2 = \mathbb{Z} \times \mathbb{Z}$, and G_3 as the HNN-extension of G_2

that switches the Z -factors, the corresponding space is a Seifert 3-manifold which is a locally trivial fibration over a circle with the fiber homeomorphic to a torus such that the monodromy (along the base circle) is a homeomorphism of the fiber that switches the meridian and the parallel of the fiber.

Choose a simplicial complex K with $\pi_1(K) = G_3$, create a covering $p: L \rightarrow K$ with $\pi_1(L) = G_2$ and $\pi_1(p)$ realizing inclusion $G_2 \rightarrow G_3$. Similarly, create a covering $q: M \rightarrow L$ with $\pi_1(M) = G_1$ and $\pi_1(q)$ realizing inclusion $G_1 \rightarrow G_2$. Notice we can give L and M structures of simplicial complexes (with resulting uniform structures generated by simplicial metrics) so that both p and q are simplicial maps. Obviously, $p \circ q$ is a simplicial covering map. However, it cannot be realized as a result of an equi-continuous action of any group G on M . Indeed, as G_1 is not normal in G_3 there is a loop α in K with two lifts β and γ (originating at different points x and y of K , obviously) such that β is a loop and γ is not a loop. Choose $g \in G$ so that $g \cdot x = y$. Notice $g \cdot \alpha$ and γ are two different lifts of the same loop, a contradiction.

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